Graded n-secondary submodules

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Throughout this work all rings are commutative with non-zero unity, unless it is expressly stated that R can be without unity. We first remind the dual of some classical graded concepts and then the dual of their extended graded notions is studied.
A proper graded submodule N of a G-graded R-module M is said to be graded prime (primary) if $a_g \in h(R)$ and $x_h \in h(M)$ with $a_g x_h \in N$ implies $a_g \in (N :_R M)$ $(a_g \in \sqrt{(N :_R M)})$ or $x_h \in N$. Also, a proper graded ideal I of R is a graded prime (primary) ideal if I is a graded prime (primary) submodule of the graded R-module R. If I is a graded ideal of R, then $G\text{-}rad(I) = \{a \in R \mid a_g \in \sqrt{I}\}$.

Lemma 1

Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. Then the following statements are equivalent.

(1) N is a graded prime (primary) submodule of M.

(2) For every $a_g \in h(R)$ and $x \in M$, $a_g x \in N$ implies $a_g \in (N :_R M)$ ($a_g \in \sqrt{(N :_R M)}$) or $x \in N$.

(3) For every $a \in R$ and $x_h \in h(M)$, $a x_h \in N$ implies $a \in (N :_R M)$ ($a \in G\text{-}rad(N :_R M)$) or $x_h \in N$.

From a functional point of view, a proper graded submodule N of a graded R-module M is a graded primary submodule, if for each $a_g \in h(R)$ the graded homomorphism $a_g. : \frac{M}{N} \to \frac{M}{N}$, that operates by multiplication, is either injective or nilpotent by the above lemma.

Let R be a G-graded ring and M a graded R-module. We say that a non-zero graded submodule N of M is graded secondary if for each $a_g \in h(R)$ the function $a_g. : N \to N$, that operates by multiplication, is either surjective or nilpotent, i.e., for every $a_g \in h(R)$ either $a_g N = N$ or $a_g \in \sqrt{(Ann_R(N)}$. Note that this concept is a dual notion of graded primary submodules in a certain sense as follows. We say that M is a graded secondary R-module if M is a graded secondary submodule of itself. Also, we say that M is a graded primary R-module if zero is a graded primary submodule of M. So the concept of graded secondary R-modules is just the dual notion of graded primary R-modules, see [9].

Let n be a positive integer. A proper graded submodule N of a graded R-module M is said to be graded n-absorbing (primary) if whenever $a_{g_1} \ldots a_{g_n} x_g \in N$ implies $a_{g_1} \ldots a_{g_n} \in (N :_R M)$ ($a_{g_1} \ldots a_{g_n} \in \sqrt{(N :_R M)}$) or $a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} x_g \in N$ for some $1 \leq i \leq n$, where $a_{g_1}, \ldots, a_{g_n} \in h(R)$, $x_g \in h(M)$ and $a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} x_g$ means $a_{g_1} \ldots a_{g_{i-1}} a_{g_{i+1}} \ldots a_{g_n} x_g$. So a graded 1-absorbing primary submodule is exactly a graded primary submodule. Also, a proper graded ideal I of R is called a graded n-absorbing (primary) ideal if I is a graded n-absorbing (primary) submodule of the graded R-module R, see [13].

In this work, we present a functional equivalent condition to the concept of graded n-absorbing primary submodules. So we can specify a dual notion of this concept in a certain sense, which we call that graded n-secondary submodules.
Next, a trivial extension of Lemma 1 is presented. Throughout, we consider the part (2) of Lemma 2 as the definition of graded n-absorbing (primary) submodules.

Lemma 2

Let n be a positive integer, R a G-graded ring, M a graded R-module and N a graded submodule of M. Then the following statements are equivalent.

(1) N is a graded n-absorbing (primary) submodule of M.

(2) For every $a_{g_1}, \ldots, a_{g_n} \in h(R)$ and $x \in M$, $a_{g_1} \ldots a_{g_n} x \in N$ implies $a_{g_1} \ldots a_{g_n} \in (N :_R M)$ ($a_{g_1} \ldots a_{g_n} \in \sqrt{(N :_R M)}$) or $a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} x \in N$ for some $1 \leq i \leq n$.

(3) For every $a_1, \ldots, a_n \in R$ and $x_g \in h(M)$, $a_1 \ldots a_n x_g \in N$ implies $a_1 \ldots a_n \in (N :_R M)$ ($a_1 \ldots a_n \in G\text{-}rad(N :_R M)$) or $a_1 \ldots \widehat{a}_i \ldots a_n x_g \in N$ for some $1 \leq i \leq n$.

Next, we present a functional method, which gives us an equivalent condition to the concept of graded n-absorbing primary submodules. For this, we need some new definitions as follows.

Definition

Let R be a G-graded ring, M a graded R-module, N a proper graded submodule of M, n a positive integer, $a_{g_1}, \ldots, a_{g_n} \in h(R)$ and the graded homomorphism $a^*_{g_i} : a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} \frac{M}{N} \to \frac{M}{N}$, defined by $a^*_{g_i}(a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n}(x + N)) = a_{g_1} \ldots a_{g_n} x + N$ for every $x \in M$, where $1 \leq i \leq n$. Then
(1) We say that the family $\{a^*_{g_i} \mid 1 \leq i \leq n\}$ is injective if $a_{g_1} \ldots a_{g_n} x \in N$ implies $a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} x \in N$ for some $1 \leq i \leq n$, where $x \in M$.
(2) We say that the family $\{a^*_{g_i} \mid 1 \leq i \leq n\}$ is nilpotent, if $a^*_{g_1} o \ldots o a^*_{g_n}$ is a nilpotent function.

Theorem

Let R be a G-graded ring, M a graded R-module and n a positive integer. A proper graded submodule N of M is a graded n-absorbing primary submodule of M if and only if the family $\{a_{g_i}^* \mid 1 \leq i \leq n\}$ of graded homomorphisms is either injective or nilpotent for every $a_{g_1}, \ldots, a_{g_n} \in h(R)$.

Definition

Let n be a positive integer and R a G-graded ring (not necessarily with unity). We say that a non-zero graded submodule N of a graded R-module M is a graded n-secondary submodule, if for every $a_{g_1}, \ldots, a_{g_n} \in h(R)$ there exists an $1 \leq i \leq n$ such that the graded homomorphism $a_{g_i}^{**} : N \to a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} N$, defined by $a_{g_i}^{**}(x) = a_{g_1} \ldots a_{g_n} x$ for every $x \in N$, is either surjective or nilpotent. So every graded 1-secondary submodule is just a graded secondary submodule.

We say that a graded R-module is a graded n-secondary R-module if M is a graded n-secondary submodule of itself. So the notion of graded n-secondary R-modules is just the dual notion of graded n-absorbing primary R-modules. Therefore, it can be concluded that the concept of graded n-secondary R-modules is just the dual notion of graded n-absorbing primary R-modules. Hence we can say that the notion of graded n-secondary submodules is the dual notion of graded n-absorbing primary submodules (in a certain sense).

Let R be a G-graded ring, n, m two positive integers with $n \leq m$ and M a graded R-module. Clearly, every graded n-secondary submodule of M is graded m-secondary. Next, we show that the converse is not true in general.

Example

Let $F = \mathbb{Z}_2$, G an arbitrary group, R the trivial G-graded polynomial ring $F[x]$, $S = Rx$, J the graded ideal generated by $\{x + x^2, x + x^3, \ldots\}$ of R and $M = \frac{R}{J}$ as a graded S-module. We claim that M is not a graded secondary S-module, while it is a graded 2-secondary S-module.
If there exists a positive integer m such that $x^m \in Ann_S(M)$, then $x^m \in J$. So there exists a positive integer k and non-zero $g_1, \ldots, g_k \in h(R)$ such that $x^m = g_1(x + x^{r_1}) + \ldots + g_k(x + x^{r_k})$, where $r_1, \ldots, r_k$ are positive integers with $r_1 < r_2 < \ldots < r_k$.

Note that the number of x^m's in the right side of the equality is an odd number and the number of summands of each of $g_i(x + x^{r_i})$ is an even number. Without loss of generality, we can assume that there exists an odd number $1 \leq l \leq k$ such that $x^m \in g_j(x + x^{r_j})$ for every $1 \leq j \leq l$. So $0 = [g_{r_1}(x + x^{r_1}) - x^m] + \ldots + [g_{r_l}(x + x^{r_l}) - x^m] + g_{r_{l+1}}(x + x^{r_{l+1}}) + \ldots + g_{r_k}(x + x^{r_k})$, where the number of summands in the right side is an odd number and so the number of non-zero summands in the right side is also an odd number. Hence there exists an odd number n such that $0 = x^{\gamma_1} + \ldots + x^{\gamma_n}$, where $\gamma_1 < \gamma_2 < \ldots < \gamma_n$ which is impossible. Therefore, $x \notin \sqrt{(Ann_S(M))}$.

We claim that $M \neq xM$. If not, there exists a $g \in h(R)$ such that $1 - xg \in J$ which is impossible. Because the elements of J are coefficients of x.

Now, we show that for each $f_1, f_2 \in h(S)$ either $f_1 f_2 M = f_1 M$ or $f_1 f_2 M = f_2 M$. Since $f_1, f_2 \in S$, there exist $g_1, g_2 \in h(R)$ such that $f_1 = xg_1$ and $f_2 = xg_2$. If the number of summands of g_1 is an even number, then $f_1 f_2 M = g_1 g_2 x^2 M = g_1 x M = f_1 M = 0$. Because for every positive integers γ, δ such that $1 \leq \gamma \leq \delta$, $x^\gamma + x^\delta = x^{\gamma-1}(x + x^{\delta-\gamma+1}) \in J$. Similarly, if the number of summands of f_2 is an even number, then $f_1 f_2 M = f_2 M = 0$. Now, suppose that the number of summands of each of f_1, f_2 is an odd number.

Let $l \in h(R)$. We claim that there exists an $l_1 \in h(R)$ such that $g_1xl - g_1g_2x^2l_1 \in J$. If the number of summands of l is an odd number, then put $l_1 = 1$. So the number of summands of $g_1xl - g_1g_2x^2l_1$ is an even number and so $g_1xl - g_1g_2x^2l_1 \in J$, as explained above. If the number of summands of l is an even number, then put $l_1 = 1 + x$. Hence the number of summands of each of $g_1xl, g_1g_2x^2l_1$ is an even number. Thus $g_1xl - g_1g_2x^2l_1 \in J$ and so $f_1M \subseteq f_1f_2M$. Hence $f_1M = f_1f_2M$. Thus either f_1^{**} is surjective or f_2^{**} is so.

Theorem

Let R be a G-graded ring, n a positive integer and M a finitely generated graded R-module. If M is a graded n-secondary R-module, then M is a graded n-absorbing primary R-module.

Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is said to be a graded completely irreducible submodule, if $N = \cap_{i \in I} N_i$ implies $N = N_i$ for some $i \in I$, where $\{N_i \mid i \in I\}$ is a family of graded submodules of M. It is easy to show that every proper graded submodule K of M is an intersection of graded completely irreducible submodules of M.

Let R be a G-graded ring and N a non-zero graded submodule of a graded R-module M. It is not difficult to verify that N is a graded secondary submodule of M if and only if for every $a_g \in h(R)$ and every graded completely irreducible submodule K of M, $a_g N \subseteq K$ implies either $a_g \in \sqrt{Ann_R(N)}$ or $N \subseteq K$. Next, this result is extended and an equivalent condition to the concept of graded n-secondary submodules is presented.

Theorem

Let n be a positive integer, R a G-graded ring, M a graded R-module and N a non-zero graded submodule of M. Then N is a graded n-secondary submodule of M if and only if for every $a_{g_1}, \ldots, a_{g_n} \in h(R)$ and every graded completely irreducible submodules $K_1, \ldots, K_n$ of M, $a_{g_1} \ldots a_{g_n} N \subseteq \cap_{j=1}^{n} K_j$ implies $a_{g_1} \ldots a_{g_n} \in \sqrt{Ann_R(N)}$ or $a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} N \subseteq \cap_{j=1}^{n} K_j$ for some $1 \leq i \leq n$.

Definition

We say that a non-zero graded submodule N of a graded R-module M is a graded n-absorbing secondary (graded strongly n-absorbing secondary) submodule, if whenever $a_{g_1} \dots a_{g_n} N \subseteq K$ implies $a_{g_1} \dots a_{g_n} \in \sqrt{Ann_R(N)}$ or $a_{g_1} \dots \widehat{a_{g_i}} \dots a_{g_n} N \subseteq K$ for some $1 \leq i \leq n$, where $a_{g_1}, \dots, a_{g_n} \in h(R)$ and K is a graded completely irreducible submodule (a graded submodule) of M. For general modules see [2,5].

Next, we show that the concepts of graded n-secondary and graded strongly n-absorbing secondary submodules are equivalent, while these are not equivalent to the concept of graded n-absorbing secondary submodules.

Theorem

Let n be a positive integer, R a G-graded ring, M a graded R-module and N a non-zero graded submodule of M. Then the following statements are equivalent.

(1) N is a graded n-secondary submodule of M.

(2) for every $a_{g_1}, \ldots, a_{g_n} \in h(R)$ and every graded submodule K of M, $a_{g_1} \ldots a_{g_n} N \subseteq K$ implies $a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} N \subseteq K$ for some $1 \leq i \leq n$ or $a_{g_1} \ldots a_{g_n} \in \sqrt{Ann_R(N)}$.

(3) for every $a_{g_1}, \ldots, a_{g_n} \in h(R)$, $a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} N = a_{g_1} \ldots a_{g_n} N$ for some $1 \leq i \leq n$ or $a_{g_1} \ldots a_{g_n} \in \sqrt{Ann_R(N)}$.

Example

Let $n \geq 2$ be a positive integer, G an arbitrary group and R the trivial G-graded ring $\mathbb{Z}$, $p_{g_1} = 2, p_{g_2}, \ldots, p_{g_{n+2}}$ distinct positive prime integers, $s = p_{g_1} \ldots p_{g_{n+2}}$ and $M = \mathbb{Z}_s$ as a graded R-module. Clearly, every graded submodule of M is of the form $p_{g_1}^{\alpha_1} p_{g_2}^{\alpha_2} \ldots p_{g_{n+2}}^{\alpha_{n+2}} \mathbb{Z}_s$, where $0 \leq \alpha_j \leq 1$ for every $1 \leq j \leq n+2$. Let $T_k = p_{g_k} \mathbb{Z}_s$ for every $1 \leq k \leq n+2$. It is not difficult to verify that the only graded completely irreducible submodules of M are $T_1, \ldots, T_{n+2}$.

Let $a_{h_1}, a_{h_2} \in h(R)$, with $a_{h_1} a_{h_2} T_1 \subseteq T_k$ for some $1 \leq k \leq n+2$. So $p_{g_k} \mid a_{h_1}$ or $p_{g_k} \mid a_{h_2}$. Thus either $a_{h_1} T_1 \subseteq T_k$ or $a_{h_2} T_1 \subseteq T_k$. Therefore, T_1 is a graded 2-absorbing secondary and so a graded n-absorbing secondary submodule of M.
Evidently, $p_{g_2} \dots p_{g_{n+1}} T_1 \subseteq \cap_{j=2}^{n+1} T_j$. But there is no $2 \leq i \leq n+1$ such that $p_{g_2} \dots \widehat{p_{g_i}} \dots p_{g_{n+1}} T_1 \subseteq \cap_{j=2}^{n+1} T_j \subseteq T_i$, because $p_{g_i} \not| p_{g_1} \dots \widehat{p_{g_i}} \dots p_{g_{n+1}}$. Also, $p_{g_2} \dots p_{g_{n+1}} \notin \sqrt{Ann_R(T_1)}$. Therefore, T_1 is not a graded n-secondary submodule of M and so it is not a graded strongly n-absorbing secondary submodule of M by the above theorem.

Theorem

Let R be a G-graded ring, M a graded R-module and n a positive integer. If N is a graded n-secondary submodule of M, then $Ann_R(N)$ is a graded n-absorbing primary ideal of R.

Let R be a G-graded ring and n a positive integer. Clearly, if I is a graded n-absorbing primary ideal of R, then $\sqrt{I}$ is a graded n-absorbing ideal of R.

Corollary

Let R be a G-graded ring, M a graded R-module and n a positive integer. If N is a graded n-secondary submodule of M, then $\sqrt{Ann_R(N)}$ is a graded n-absorbing ideal of R.

Let R be a G-graded ring. If N is a graded n-secondary submodule of a graded R-module M with $p = \sqrt{Ann_R(N)}$, then we say that N is a graded p-n-secondary submodule of M.

Definition

Let R be a G-graded ring (not necessarily with unity) and n a positive integer. We say that a graded R-module M is graded n-divisible, if for every $x \in M$ and every $a_{g_1}, \ldots, a_{g_n} \in h(R)$ with $(a_{g_1} \ldots a_{g_n})^m \neq 0$ for every positive integer m there exists an $1 \leq i \leq n$ and a $y \in M$ such that $a_{g_1} \ldots a_{g_n} y = a_{g_1} \ldots \widehat{a_{g_i}} \ldots a_{g_n} x$. Clearly, if M is a non-zero graded 1-divisible R-module, then $Ann_R(M) \cap h(R) = 0$.

Example

Let G be a group, I an index set, R_i a G-graded integral domain, K_i the field of fractions of R_i for every $i \in I$ and R the G-graded ring $\oplus_{i \in I} R_i$. Then $K = \oplus_{i \in I} K_i$ is a graded n-divisible R-module for every positive integer n.

Theorem

Let R be a G-graded ring, n a positive integer, M a graded R-module and N a non-zero graded submodule of M with $p = \sqrt{(Ann_R(N)}$. Then the following statements are equivalent.
(1) N is a graded p-n-secondary submodule of M.
(2) N is a graded n-divisible $\frac{R}{p}$-module.

Let R be a G-graded ring and n, m two positive integers with $m \leq n$. Evidently, every graded m-divisible R-module is a graded n-divisible R-module. Nevertheless, we show that the converse is not true in general.

Example

Let R be the $\mathbb{Z}$-graded polynomial ring $F[x]$, where F is a field, m, n two positive integers with $m < n$, $T = \oplus_{i \in \mathbb{Z}} T_i$, where $T_i = R_i$ for every non-zero integer i, $T_0 = 0$, $S = \oplus_{i \in \mathbb{Z}} S_i$, where $S_i = R_i$ for every $i \geq n$, $S_i = 0$ for every $i < n$ and $M = \frac{T}{S}$ as a T-module. Let $f_{g_1}, \ldots, f_{g_n} \in h(T)$ with $(f_{g_1} \ldots f_{g_n})^t \neq 0$ for every positive integer t. If $f + S \in M$, then $f_{g_1} \ldots \widehat{f_{g_i}} \ldots f_{g_n} f - f_{g_1} \ldots f_{g_n} x \in S$ for every $1 \leq i \leq n$. Thus M is an n-divisible T-module.
Let $f_{g_1} = \ldots = f_{g_m} = x$. Then there is no $f + S \in M$ such that $x^m f + S = x^{m-1}(x + S)$. Therefore, M is not an m-divisible T-module.

Recall that a proper graded ideal m of R is said to be a maximal graded ideal if $m \subseteq I \subseteq R$ implies $I = m$ or $I = R$, where I is a graded ideal of R. We denote the set of all maximal graded ideals of R by $g\text{-}Max(R)$.

Proposition

Let R be a G-graded ring, n a positive integer and M a graded R-module. If N is a graded submodule of M with $Ann_R(N) = p \in g\text{-}Max(R)$, then N is a graded p-n-secondary submodule of M.

Theorem

Let $n \geq 2$ be a positive integer, R a G-graded ring, M a graded R-module and N a graded n-secondary submodule of M. Then the following statements hold.

(1) If K is a graded submodule of M with $N \not\subseteq K$, then $(K :_R N)$ is a graded n-absorbing primary ideal of R.

(2) If $G\text{-}rad(Ann_R(N))$ is a graded $(n-1)$-absorbing ideal of R, then $\sqrt{(K :_R N))}$ is a graded $(n-1)$-absorbing ideal of R for every graded submodule K of M with $N \not\subseteq K$.

(3) $G\text{-}rad(\cap_{i=1}^{n} L_i :_R N))$ is a graded $(n-1)$-absorbing ideal of R for every family of graded completely irreducible submodules $\{L_1, \ldots, L_n\}$ of M with $N \not\subseteq \cap_{i=1}^{n} L_i$ if and only if $G\text{-}rad(Ann_R(N))$ is a graded $(n-1)$-absorbing ideal of R.

Theorem

Let R be a G-graded ring, n a positive integer, M a graded R-module and N a graded n-secondary submodule of M.Then $a_g N$ is a graded n-secondary submodule of M for every $a_g \in h(R) \setminus Ann_R(N)$.

Theorem

Let n be a positive integer, R a G-graded ring and $f : M_1 \to M_2$ a graded R-module homomorphism. If N is a graded n-secondary submodule of M_1 with $f(N) \neq 0$, then $f(N)$ is a graded n-secondary submodule of M_2.

Theorem

Let n be a positive integer, R a G-graded ring, M a graded R-module and L, N two graded submodules of M with $L \subset N$. If N is a graded n-secondary submodule of M, then $\frac{N}{L}$ is a graded n-secondary submodule of the graded R-module $\frac{M}{L}$.

Let m be a positive integer, R_i a G-graded ring, M_i a graded R_i-module for every $1 \leq i \leq m$, $R = R_1 \times \ldots \times R_m$ and $M = M_1 \times \ldots \times M_m$. Clearly, R is a G-graded ring with $R_g = R_{1_g} \times \ldots \times R_{m_g}$ and M a graded R-module with $M_g = M_{1_g} \times \ldots \times M_{m_g}$ for every $g \in G$, see [14]. It is easy to show that every graded submodule of the graded R-module $M = M_1 \times \ldots \times M_m$ is of the form $N = N_1 \times \ldots \times N_m$, where N_i is a graded submodule of M_i for every $1 \leq i \leq m$. In this case, we call N_i's the graded components of N.

Theorem

Let m, n be two positive integers and R, M the same as in above. If N is a graded n-secondary submodule of M, then every non-zero graded component N_j of N is a graded n-secondary submodule of the graded R_j-module M_j, where $1 \leq j \leq m$.

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